REVISED SOLUTION OF PROBLEM EIGHTY NINE.

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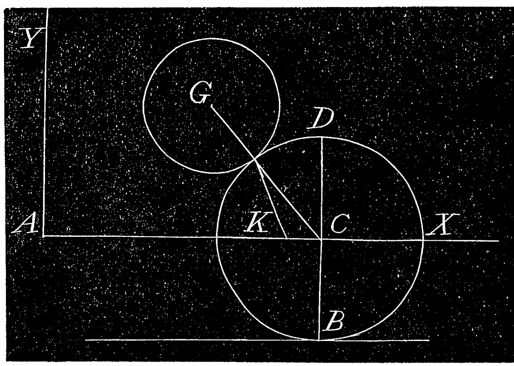
Problem.—"A sphere, radius r, *rolls* down the surface of another sphere of the same material, radius R, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. Determine the motion of the spheres, the point of separation and the equation of the curve described by the center of the upper sphere."

LET x, y, be the coordinates, referred to the axes AX, AY, of the centre G of the upper sphere at time t after motion begins, $\psi =$ the angle GCX which the line of centres CG makes with the axis of X, $\alpha =$ initial value of ψ, $a =$ initial value of x; $\theta_1 - \alpha =$ angular rotation of lower sphere about its centre, $R(\theta_1 - \alpha) =$ horizontal distance moved by C the centre of the lower sphere; then



$$x = a + (r + R)(\cos\psi - \cos\alpha) - R(\theta_1 - \alpha), \quad \ldots\ldots (1)$$

$$y = (r + R)\sin\psi, \quad \ldots\ldots (2)$$

$$\frac{R(\psi - \alpha) - R(\theta_1 - \alpha)}{r} + \psi - \alpha = \frac{(r + R)\psi - R\theta_1}{r} - \alpha = \theta_2, \ldots (3)$$

$=$ angular rotation of upper sphere,

$$dx^2 + dy^2 = R^2 d\theta_1^2 + 2R(r + R)\sin\psi\, d\psi\, d\theta_1 + (r + R)^2 d\psi^2 \ldots (4)$$

There being no loss of vis viva in perfect rolling,

$$2m'g(r + R)(\sin\alpha - \sin\psi) = m'\frac{dx^2 + dy^2}{dt^2} + \frac{2}{5}m'r^2\frac{d\theta_2^2}{dt^2} + \frac{7}{5}mR^2\frac{d\theta_1^2}{dt^2}$$

$$= \frac{7}{5}\left(m + m'\right)R^2\frac{d\theta_1^2}{dt^2} + \frac{7}{5}m'\left(r + R\right)^2\frac{d\psi^2}{dt^2} + 2m'R[r + R]\left(\sin\psi - \frac{2}{5}\right)\frac{d\psi\, d\theta_1}{dt^2} \quad \ldots (5)$$

Let $F =$ the resultant action between the spheres at the point of contact H, $\varphi =$ the angle CHK which its direction makes with the normal HC, then the equations of motion of the upper sphere are

$$F\cos(\psi - \varphi) = m'\frac{d^2x}{dt^2}, \quad \ldots\ldots (6)$$

$$F\sin(\psi - \varphi) - m'g = \frac{d^2y}{dt^2}, \quad \ldots\ldots (7)$$

$$\frac{Fr\sin\varphi}{\frac{2}{5}m'r^2}=\frac{F\sin\varphi}{\frac{2}{5}m'r}=\frac{d^2\theta_2}{dt^2}, \quad \text{. . . . . (8)}$$

and for rotation of lower sphere about lower point of contact B,

$$\frac{F\cos(\psi-\varphi)\times R(1+\sin\psi)-F\sin(\psi-\varphi)\times R\cos\psi}{\frac{7}{5}mR^2}$$

$$=F\times\frac{\cos(\psi-\varphi)+\sin\varphi}{\frac{7}{5}mR}=\frac{d^2\theta_1}{dt^2}. \quad \text{. . . (9)}$$

Eliminating between (6), (8), (9),

$$m'\frac{d^2x}{dt^2}+\frac{2}{5}m'r\frac{d^2\theta_2}{dt^2}=\frac{7}{5}mR\frac{d^2\theta_1}{dt^2}. \quad \text{. . . . . (10)}$$

Integrating and observing that the velocities begin at the same time,

$$m'\frac{dx}{dt}+\frac{2}{5}m'r\frac{d\theta_2}{dt}=\frac{7}{5}mR\frac{d\theta_1}{dt}. \quad \text{. . . . . (11)}$$

Eliminating by (1) and (3)

$$\frac{d\theta_1}{dt}=\frac{m'(r+R)\left(\frac{2}{5}-\sin\psi\right)\frac{d\psi}{dt}}{\frac{7}{5}R(m+m')}. \quad \text{. . . . . (12)}$$

Integrating between limits,

$$\theta_1-a=\frac{\left[\frac{2}{5}(\psi-a)+\cos\psi-\cos a\right]m(r+R)}{\frac{7}{5}R(m+m')}. \quad \text{. . . . . .(13)}$$

Hence by (1) and (2)

$$x=a+(r+R)(\cos\psi-\cos a)-\frac{m'(r+R)\left[\frac{2}{5}(\psi-a)+\cos\psi-\cos a\right]}{\frac{7}{5}(m+m')}$$

$$=a+\frac{[7(m+m')-5m'](r+R)(\cos\psi-\cos a)-2m'(r+R)(\psi-a)}{7(m+m')},$$

$$7(m+m')(x-a)=(7m+2m')\left\{\left[(r+R)^2-y^2\right]^{\frac{1}{2}}-(r+R)\cos a\right\}$$

$$-2m'(r+R)\left(\sin^{-1}\frac{y}{r+R}-a\right), \quad \text{. . . (14)}$$

which is the equation of the required curve.

Changing signs of (14), it is seen that as y decreases, $a-x$ increases, therefore the motion of the upper sphere is toward the origin A as represented in the figure; and from (12) and (13) the motion of the lower sphere is, for values of a not much greater than $\frac{1}{2}\pi$, at first, from the origin A, until $\sin\psi=\frac{2}{5}$, when it comes to rest, where its motion is reversed, and returns to its original position when $\frac{2}{5}(\psi-a)+\cos\psi-\cos a=0$.

At the time of separation, since the component of F normal to the two surfaces equals zero, equation (9) becomes

$$\frac{F\sin\phi(1+\sin\phi)}{\frac{7}{5}mR} = \frac{d^2\theta_1}{dt^2}, \ldots\ldots\ldots\ldots (15)$$

hence by (8)

$$\frac{7}{5}mR\frac{d^2\theta_1}{dt^2} = \frac{2}{5}m'r\frac{d^2\theta_2}{dt^2} = \frac{2}{5}m'\left[(r+R)\frac{d^2\psi}{dt^2} - R\frac{d^2\theta_1}{dt^2}\right],$$

from which

$$\frac{d^2\theta_1}{dt^2} = \frac{2m'(r+R)(1+\sin\psi)\frac{d\psi^2}{dt^2}}{R[7m+2m'(1+\sin\psi)]}\ldots\ldots\ldots (16)$$

From (16) I conclude, that at the point of separation the tendency of the forces acting on the lower sphere is to turn it in the same direction as the rotation of the centre of the upper about the lower.

From (16) and (12)

$$\frac{d^2\theta_1}{dt^2} = \frac{2m'(r+R)(1+\sin\psi)}{R[7m+2m'(1+\sin\psi)]} = \frac{m'(r+R)\left([\frac{2}{5}-\sin\psi]\frac{d^2\psi}{dt^2} - \cos\psi\frac{d\psi^2}{dt^2}\right)}{\frac{7}{5}R(m+m')},$$

hence

$$\frac{d^2\psi}{dt^2} = \frac{[7m+2m'(1+\sin\psi)]\cos\psi + \frac{7}{5}(m+m')(1+\sin\psi)}{(\frac{2}{5}-\sin\psi)[7m+2m'(1+\sin\psi)]}\times\frac{d\psi^2}{dt^2}\ldots (17)$$

And from (5) and (12),

$$\frac{d\psi^2}{dt^2} = \frac{\frac{14}{5}g(m+m')(\sin\alpha-\sin\psi)}{(r+R)[\frac{49}{25}(m+m') - m'(\frac{2}{5}-\sin\psi)^2]}\ldots\ldots (18)$$

Differentiating (18),

$$\frac{d^2\psi}{dt^2} = -\frac{\frac{49}{25}(m+m') - m'(\frac{2}{5}-\sin\psi) + 2m'(\sin\alpha-\sin\psi)(\frac{2}{5}-\sin\psi)}{[\frac{49}{25}(m+m') - m'(\frac{2}{5}-\sin\psi)^2]^2} \times\frac{\frac{7}{5}g(m+m')}{r+R}\times\cos\psi;$$

from which by (17) and (18)

$$\frac{[7m+2m'(1+\sin\psi)]\cos\psi + \frac{7}{5}(m+m')(1+\cos\psi)}{(\frac{2}{5}-\sin\psi)[7m+2m'(1+\sin\psi)]}\times(\sin\alpha-\sin\psi)$$
$$= -\cos\psi\times\frac{\frac{49}{25}(m+m') - m'(\frac{2}{5}-\sin\psi) + 2m'(\sin\alpha-\sin\psi)(\frac{2}{5}-\sin\psi)}{\frac{49}{25}(m+m') - m'(\frac{2}{5}-\sin\psi)^2},$$

which is the equation for the point of separation.

[The foregoing revised solution of problem 89 by Mr. Adcock, is published by his request and entirely at his expense; an extra half sheet having been added to make room for it.]